Permanents, Derangements, Arrangements

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```
In [40]: from sympy import *
         from sympy.utilities.iterables import subsets
         init_printing()
         from IPython.display import *

In [41]: s,t,x,u,v=symbols('s t x u v')
         var('alpha')
```

Out[41]:

$$\alpha$$

```
In [42]: def perm(A):
             if A.shape==(1,1):
                 return A[0,0]
             else:
                 n=A.shape[0]
                 return sum([A[0,i]*perm(A.minorMatrix(0,i)) for i in range(n)])

In [43]: def derange(n):
             if n==0:
                 return 1
             else:
                 return n*derange(n-1)+(-1)**n

In [44]: for d in range(1,6):
             A=t*ones(d,d)+s*eye(d)
             display(expand(simplify(perm(A))))
```

$$s+t$$

$$s^2+2st+2t^2$$

$$s^3+3s^2t+6st^2+6t^3$$

$$s^4+4s^3t+12s^2t^2+24st^3+24t^4$$

$$s^5+5s^4t+20s^3t^2+60s^2t^3+120st^4+120t^5$$

```
In [45]: [perm(ones(d,d)-eye(d)) for d in range(1,9)]
```

Out[45]:

$$[0,\quad 1,\quad 2,\quad 9,\quad 44,\quad 265,\quad 1854,\quad 14833]$$

In [46]:
```
y=[derange(i) for i in range(21)]
y
```

Out[46]:

$$[1,\ 0,\ 1,\ 2,\ 9,\ 44,\ 265,\ 1854,\ 14833,\ 133496,\ 1334961,\ 14684570,\ 176214841,\ 2290792932,\ 32071101049,\ 481066515734,\ 7697064251745,\ 130850092279664,\ 2355301661033953,\ 44750731559645106,\ 895014631192902121]$$

In [47]:
```
def arrange(n):
    if n==0:
        return 1
    else:
        return n*arrange(n-1)+1
```

In [48]:
```
[perm(ones(d,d)+eye(d)) for d in range(1,9)]
```

Out[48]:

$$[2,\quad 5,\quad 16,\quad 65,\quad 326,\quad 1957,\quad 13700,\quad 109601]$$

In [49]:
```
y=[arrange(i) for i in range(21)]
y
```

Out[49]:

$$[1,\ 2,\ 5,\ 16,\ 65,\ 326,\ 1957,\ 13700,\ 109601,\ 986410,\ 9864101,\ 108505112,\ 1302061345,\ 16926797486,\ 236975164805,\ 3554627472076,\ 56874039553217,\ 966858672404690,\ 17403456103284421,\ 330665665962404000,\ 6613313319248080001]$$

In [50]:
```
def h(a,b,s,t):
    return integrate(exp(-x)*(s+t*x)**a*(t*x)**b,(x,0,oo))
```

In [51]:
```
# This gives arrangements
[h(k,0,1,1) for k in range(21)]
```

Out[51]:

$$[1,\ 2,\ 5,\ 16,\ 65,\ 326,\ 1957,\ 13700,\ 109601,\ 986410,\ 9864101,\ 108505112,\ 1302061345,\ 16926797486,\ 236975164805,\ 3554627472076,\ 56874039553217,\ 966858672404690,\ 17403456103284421,\ 330665665962404000,\ 6613313319248080001]$$

In [52]:
```
# This give derangements
[h(k,0,-1,1) for k in range(21)]
```

Out[52]:

$$[1,\ 0,\ 1,\ 2,\ 9,\ 44,\ 265,\ 1854,\ 14833,\ 133496,\ 1334961,\ 14684570,\ 176214841,\ 2290792932,\ 32071101049,\ 481066515734,\ 7697064251745,\ 130850092279664,\ 2355301661033953,\ 44750731559645106,\ 895014631192902121]$$

In [53]:
```
k=2
Z=list(subsets(range(d),k))
Z
```

Out[53]:

$$[(0,\ 1),\ (0,\ 2),\ (0,\ 3),\ (0,\ 4),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 3),\ (2,\ 4),\ (3,\ 4)]$$

```
In [54]: def zpow(E,k):
             d=E.shape[0]
             N=binomial(d,k)
             Z=list(subsets(range(d),k))
             return Matrix(N,N,lambda i,j: expand(perm(E.extract(Z[i],Z[j]))))
```

```
In [55]: d=4
         A=t*ones(d,d)+s*eye(d)
         A
```

Out[55]:

$$\begin{bmatrix} s+t & t & t & t \\ t & s+t & t & t \\ t & t & s+t & t \\ t & t & t & s+t \end{bmatrix}$$

```
In [56]: l=2
         Al=zpow(A,l)
         Al
```

Out[56]:

$$\begin{bmatrix} s^2+2st+2t^2 & st+2t^2 & st+2t^2 & st+2t^2 & st+2t^2 & 2t^2 \\ st+2t^2 & s^2+2st+2t^2 & st+2t^2 & st+2t^2 & 2t^2 & st+2t^2 \\ st+2t^2 & st+2t^2 & s^2+2st+2t^2 & 2t^2 & st+2t^2 & st+2t^2 \\ st+2t^2 & st+2t^2 & 2t^2 & s^2+2st+2t^2 & st+2t^2 & st+2t^2 \\ st+2t^2 & 2t^2 & st+2t^2 & st+2t^2 & s^2+2st+2t^2 & st+2t^2 \\ 2t^2 & st+2t^2 & st+2t^2 & st+2t^2 & st+2t^2 & s^2+2st+2t^2 \end{bmatrix}$$

```
In [57]: n=A.shape[0]
         display([h(l-i,i,s,t) for i in range(1+min(l,n-l))])
         display([expand(simplify(s**alpha*t**(alpha+l-n)/factorial(n-l-alpha)*h(l-alpha,n-l-alpha,s,t)
         [binomial(n,alpha)-binomial(n,alpha-1) for alpha in range(1+min(l,n-l))]
```

$$\left[s^2+2st+2t^2, \quad st+2t^2, \quad 2t^2\right]$$

$$\left[s^2+6st+12t^2, \quad s^2+2st, \quad s^2\right]$$

Out[57]:

$$[1, \quad 3, \quad 2]$$

```
In [58]: Al.eigenvals()
```

Out[58]:

$$\left\{s^2 : 2, \quad s^2+2st : 3, \quad s^2+6st+12t^2 : 1\right\}$$

$$\frac{s^{\alpha}t^{\alpha+l-n}}{(n-l-\alpha)!}\,h(l-\alpha, n-l-\alpha, s, t)$$

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In [ ]:
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In [ ]:
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